Fourier Analysis

Mar 10, 2022

Review.

A cts but nowhere diff function on $\mathbb{R}$.

Let $\alpha \in (0,1)$. set

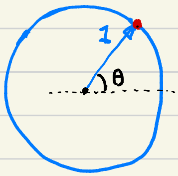
$$f_\alpha(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i 2^n x}, \quad x \in \mathbb{R}.$$

- By modifying the proof of f_α being nowhere diff slightly, one can show the real and imaginary parts of f_α are also nowhere diff. That is,

$$\sum_{n=0}^{\infty} 2^{-n\alpha} \cos(2^n x), \quad \sum_{n=0}^{\infty} 2^{-n\alpha} \sin(2^n x)$$

are nowhere diff. (Check the outline proof in the text book).

§4.4 Heat equation on the circle.



heat conduction on the circle.

$\theta \in [0, 2\pi)$

$\theta = 2\pi x, \quad x \in [0,1)$.

Let $u = u(x,t)$ be the temperature at point x and time t.

Then u satisfies

$$\frac{\partial u}{\partial t} = c \cdot \frac{\partial^2 u}{\partial x^2}, \qquad x \in [0,1), \quad t > 0.$$

By scaling the time t if necessary, we may obtain a standard heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad x \in [0,1), \quad t > 0. \qquad (*)$$

The function $u = u(x,t)$ can be extended to a function on $\mathbb{R} \times (0,\infty)$, which is 1-periodic in x.

Here we would like to find a solution u of $(*)$, in the mean time, we ask u to satisfy an initial condition

$$u(x,0) = f(x), \qquad (**)$$

We first find some special solutions $u(x,t) = A(x)B(t)$ of $(*)$.

Plugging $u = A(x)B(t)$ into $(*)$, we obtain

$$B'(t)\,A(x) = A''(x)\,B(t)$$

So

$$\frac{A''(x)}{A(x)} = \frac{B'(t)}{B(t)} = \lambda.$$

From $A''(x) - \lambda A(x) = 0$, we obtain

$$A(x) = \begin{cases} c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x} & \text{if } \lambda > 0 \\ c_1 x + c_2 & \text{if } \lambda = 0 \\ c_1 e^{i\sqrt{-\lambda}x} + c_2 e^{-i\sqrt{-\lambda}x} & \text{if } \lambda < 0 \end{cases}$$

To obtain a 1-periodic solution $A(x)$, we must have $A(x)$ be a const or

$$\underbrace{A(x) = c_1 e^{in2\pi x} + c_2 \cdot e^{-in2\pi x}}_{\text{corresponding to } \lambda = -(2\pi n)^2 = -4\pi^2 n^2} \quad \text{for some } n \in \mathbb{Z}$$

$$\frac{B'(t)}{B(t)} = -4\pi^2 n^2 \Rightarrow B(t) = c \cdot e^{-4\pi^2 n^2 t}$$

Hence our special solution should be of the form

$$u(x,t) = \left(c_1 e^{i2\pi nx} + c_2 e^{-i2\pi nx}\right) \cdot c \cdot e^{-4\pi^2 n^2 t}$$
$$= \left(a_n e^{i2\pi nx} + a_{-n} e^{-i2\pi nx}\right) e^{-4\pi^2 n^2 t}.$$

Using the superposition of these special solutions we would like to find a_n $(n \in \mathbb{Z})$ such that

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n e^{i2\pi nx} \cdot e^{-4\pi^2 n^2 t} \qquad (***)$$

Such that $\quad u(x,0) = f(x).$

When putting $t=0$ in $(***)$, we obtain

$$\sum_{n=-\infty}^{\infty} a_n e^{i2\pi nx} = f(x)$$

That is, the (LHS) is the Fourier series of f on $[0,1)$.

i.e. $\quad a_n = \hat{f}(n) = \int_0^1 f(x) e^{-2\pi inx} dx$

(In general, the Fourier series of f on $[a,b]$ is

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{i\frac{2\pi}{b-a}nx}, \qquad \text{where}$$

$$\hat{f}(n) = \frac{1}{b-a}\int_a^b f(x)\, e^{-i\frac{2\pi}{b-a}nx}\,dx\;)$$

Now the guessed solution of u satisfying $(*)$, $(**)$

is

$$u(x,t) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{2\pi i n x}\, e^{-4\pi^2 n^2 t}, \qquad x \in [0,1),\ t > 0$$

We still need to verify, under certain suitable condition on f, the above solution really satisfies both $(*)$ and $(**)$.

Proposition 1: Let f be 1-periodic function on $\mathbb{R}$. Assume that f is Riemann integrable on $[0,1)$. Then

$$u(x,t) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi i n x}\, e^{-4\pi^2 n^2 t}$$

satisfies $(*)$. Furthermore if f is cts at x_0, then $\lim_{t\to 0} u(x_0,t) = f(x_0)$.